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THE INFLUENCE OF LATERAL MASS FLUX ON FREE CONVECTION BOUNDARY LAYERS IN A SATURATED POROUS MEDIUM

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Abstract –The effects of lateral mass flux with prescribed temperature and velocity on vertical free convection boundary layers in a saturated porous medium at high Rayleigh numbers are studied analytically in this paper. Within the framework of boundary-layer theory, similarity solutions are obtained for the special case where the prescribed temperature and velocity of the fluid vary as x^{λ} and $x^{(\lambda-1)/2}$ respectively. The effects of mass flux on surface heat-transfer rate and boundary-layer thickness are shown. Application to warm water discharge along a well or fissure to an aquifer of infinite extent is discussed.

NOMENCLATURE

- A, constant defined by equation (3a);
- a, constant defined by equation (3b);
- C_p, specific heat of the convective fluid;
- f, dimensionless stream function defined by equation (10);
- f_w, lateral mass flux parameter;
- g, acceleration due to gravity;
- K, permeability of the porous medium;
- k, thermal conductivity of the porous medium;
- L, length of the source or sink;
- $\dot{m}$, mass-transfer rate;
- n, constant defined by equation (3b);
- Q, over-all surface heat-transfer rate;
- q, local heat-transfer rate;
- Ra_x, modified local Rayleigh number, $Ra_x \equiv \rho_\infty g\beta K|T_w - T_\infty|x/\mu\alpha$;
- T, temperature;
- u, Darcy's velocity in vertical direction;
- v, Darcy's velocity in horizontal direction;
- x, vertical coordinate;
- y, horizontal coordinate.

Greek symbols

- α, equivalent thermal diffusivity;
- β, coefficient of thermal expansion;
- δ, boundary-layer thickness;
- η, dimensionless similarity variable defined by equation (9);
- η_δ, value of η at the edge of boundary layer;
- θ, dimensionless temperature defined by equation (11);
- λ, constant defined by equation (3a);
- μ, viscosity of convective fluid;
- ρ, density of convective fluid;
- ψ, stream function.

Subscript

- ∞, condition at infinity;
- w, condition at the wall.

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INTRODUCTION

THE EFFECTS of blowing and suction along a vertical flat plate on free convection in air or water have been the subject of numerous investigations. Eichhorn [1] studied these effects for a class of problems where both wall temperature and the blowing or suction velocity are prescribed power functions of distance from the leading edge. Based on the boundary-layer approximations, Eichhorn shows that similarity solutions for the problem are possible if the exponents in the prescribed power functions are related in a particular manner. Sparrow and Cess [2] studied the more general problem with arbitrary values of exponents by a perturbation method. The problem was also studied by Mabuchi [3] who used an integral method.

The analogous problem of lateral injection or withdrawal of fluid along a vertical plane source or sink on free convection boundary layers in a porous medium at high Rayleigh numbers, where both the temperature distribution of the fluid along the plane source or sink (T_w) and its velocity distribution (v_w) are prescribed power functions of distance, is studied in the present paper. If the boundary-layer approximations similar to those employed by Wooding [4], McNabb [5], Cheng and Minkowycz [6], and Cheng and Chang [7] are invoked, and if the prescribed power functions are given by $T_w = T_\infty \pm Ax^{\lambda}$ and $v_w = ax^n$, it is found that similarity solutions are possible if $n = (\lambda-1)/2$. The problem has a number of important engineering and geophysical applications. For example, the residual warm water discharged from a geothermal power plant is usually disposed of through subsurface reinjection wells which can be idealized as vertical plane sources in a porous medium. If the temperature of the injected fluid differs from that of the receiving groundwater in the rock formation, the injected fluid would experience a positive or negative buoyancy force (depending on the relative temperature difference) which results in a convective movement of groundwater near the well. Similarly, convection of groundwater also occurs along the vertical fissures or cracks during the natural recharge of aquifer, whenever the temperature of the groundwater discharged from the fissures and cracks differs from that of the receiving water in the aquifer.

ANALYSIS

Figure 1 shows the problem of recharge or withdrawal of fluids along a vertical plane source or sink embedded in a saturated porous medium, where the temperature along the source or sink is given by $T_w = T_\infty \pm Ax^\lambda$ (with T_∞ denoting temperature at infinity and $A > 0$) and the discharge or withdrawal rate is given by $v_w = ax^n$ where $a > 0$ for discharge of fluid and $a < 0$ for withdrawal of fluid. If we assume that (i)

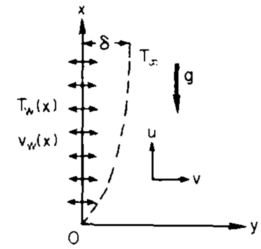

FIG. 1(a). Coordinate system for $T_w > T_\infty$.

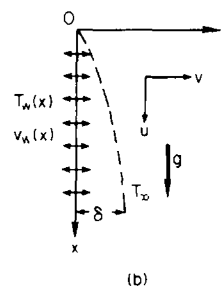

FIG. 1(b). Coordinate system for $T_w < T_\infty$.

the convective flow is due to the density difference between the source (or sink) and at infinity, (ii) the temperature of the fluid is everywhere below the boiling point, (iii) the convective fluid and the porous medium are everywhere in local thermodynamic equilibrium, (iv) properties of the fluid and the porous medium such as viscosity, thermal conductivity, thermal expansion coefficient, specific heats and permeability are constant, and (v) the Boussinesq approximation is employed, it can be shown that the governing equations with boundary layer simplifications are given by [6]

$$\frac{\partial^2 \psi}{\partial y^2} = \pm \frac{\rho_\infty \beta g K}{\mu} \frac{\partial T}{\partial y}, \tag{1}$$

$$\frac{\partial^2 T}{\partial y^2} = \frac{1}{\alpha}\left(\frac{\partial \psi}{\partial y}\frac{\partial T}{\partial x} - \frac{\partial \psi}{\partial x}\frac{\partial T}{\partial y}\right), \tag{2}$$

where the "+" and "−" signs in equation (1) are for the cases of $T_w > T_\infty$ (Fig. 1a) and $T_w < T_\infty$ (Fig. 1b) respectively. In equations (1) and (2), ρ_∞, μ and β are the density, viscosity, and the thermal expansion coefficient of the fluid; g is the gravitational acceleration; K is the permeability of the porous medium; $\alpha = k/(\rho_\infty C_p)_f$ is the equivalent thermal diffusivity with k denoting the thermal conductivity of the porous medium and $(\rho_\infty C_p)_f$ the density and specific heat of the fluid; ψ is the stream function defined in the usual manner, i.e. $u = (\partial\psi/\partial y)$ and $v = -(\partial\psi/\partial x)$ where u and v are the Darcy's velocities in the x and y directions.

For the coordinate system shown in Fig. 1, the boundary conditions are given by

$$y = 0, \quad T = T_\infty \pm Ax^\lambda, \quad v = -\frac{\partial \psi}{\partial x} = ax^n, \tag{3a, b}$$

$$y \to \infty, \quad T = T_\infty, \quad u = \frac{\partial \psi}{\partial y} = 0, \tag{4a, b}$$

where $A > 0$ and the "+" and "−" signs in equation (3a) are for Figs. 1(a) and (b), respectively. With the exception of boundary condition (3b), equations (1)–(4) are identical to the governing equations and boundary conditions for the problem of free convection about a vertical flat plate embedded in a porous medium where similarity solutions have been found [6]. It can be shown that similarity solutions to equations (1)–(4) exist if $n = (\lambda - 1)/2$, and that under such a restricted condition the governing equations can be transformed into

$$f'' - \theta' = 0, \tag{5}$$

$$\theta'' + \frac{1+\lambda}{2} f\theta' - \lambda f'\theta = 0, \tag{6}$$

with boundary conditions given by

$$\eta = 0, \quad \theta = 1, \quad f = f_w, \tag{7a, b}$$

$$\eta \to \infty, \quad \theta = 0, \quad f' = 0, \tag{8a, b}$$

where the similarity variables η, f, and θ are defined by

$$\eta = \left[\frac{\rho_\infty g\beta K|T_w - T_\infty|}{\mu\alpha x}\right]^{1/2} y, \tag{9}$$

$$\psi = \left[\frac{\alpha\rho_\infty g\beta K|T_w - T_\infty|x}{\mu}\right]^{1/2} f(\eta), \tag{10}$$

$$\theta(\eta) = \frac{T - T_\infty}{T_w - T_\infty}, \tag{11}$$

and $f_w \equiv -2a/[\alpha\rho_\infty g\beta KA/\mu]^{1/2}(1+\lambda)$ which is positive for the withdrawal of fluid and negative for the discharge of fluid. In terms of the new variables, the vertical and horizontal velocity components are given by

$$u = \frac{\rho_\infty g\beta|T_w - T_\infty|K}{\mu} f'(\eta), \tag{12}$$

and

$$v = 1/2\left[\frac{\alpha\rho_\infty g\beta K|T_w - T_\infty|}{\mu x}\right]^{1/2} [(1-\lambda)\eta f' - (1+\lambda)f]. \tag{13}$$

With the aid of boundary condition (8), equation (5) can be integrated to give

$$f' - \theta = 0, \tag{14}$$

which shows that vertical velocity and temperature have the same shape.

RESULTS AND DISCUSSION

Equations (14), (6), (7) and (8a) are the governing equations and boundary conditions for the problem, which can be integrated numerically by means of the Runge–Kutta method incorporated with the shooting technique for a systematic guessing of $\theta'(0)$. Numerical results for $f(\eta)$, $\theta'(\eta)$, $f'(\eta)$ or $\theta(\eta)$ for selected values of λ with $f_w = -1.0$ to 1.0 are shown in Figs. 2–4. It is noted that $f_w = 0$ corresponds to the case of an impermeable vertical flat plate embedded in a porous medium [6].

Figure 2 shows that the value of θ or f' decreases from 1 to 0 as η is increased from zero at different values of f_w. If the edge of the boundary layer thickness (denoted by η_δ) is defined as the value of η where θ (or f') has a value of 0.01, it follows from equation (9) that

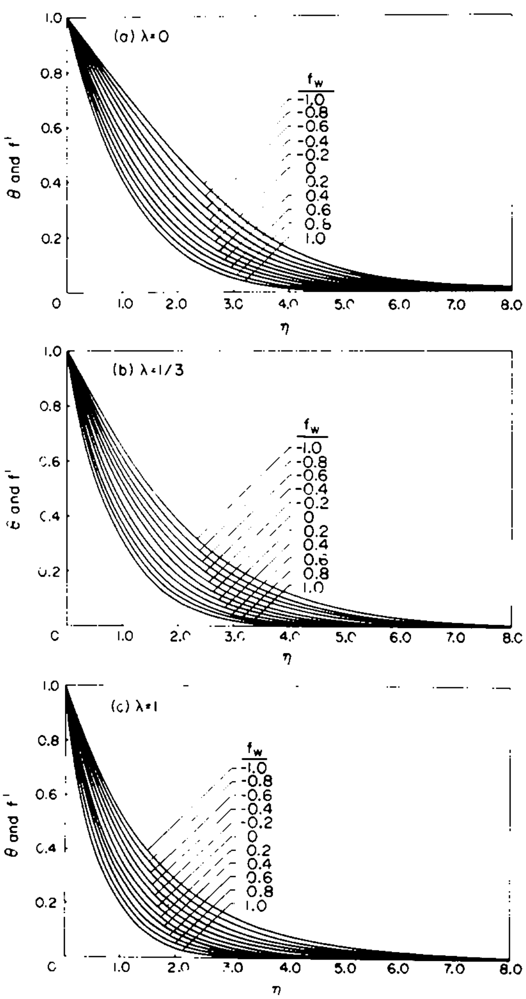

FIG. 2. Values of θ and f' vs η: (a) Uniform wall temperature distribution ($\lambda = 0$), (b) Uniform surface heat flux ($\lambda = 1/3$), and (c) Uniform wall velocity distribution ($\lambda = 1$).

the expression for the boundary-layer thickness δ is

$$\frac{\delta}{x} = \frac{\eta_\delta}{(Ra_x)^{1/2}}, \tag{15}$$

where $Ra_x = \rho_\infty g K \beta x |T_w - T_\infty| / \mu\alpha$ is the modified Rayleigh number, and the value of η_δ for selected λ is presented in Table 1 and plotted in Fig. 5, where it is shown that the boundary-layer thickness decreases as the value of f_w increases from -1.0 to 1.0. The expression for local heat flux can be shown to be

$$q(x, y) = -k\frac{\partial T}{\partial y} = kA^{3/2}\left[\frac{\rho_\infty g\beta K}{\mu\alpha}\right]^{1/2} x^{\frac{3\lambda-1}{2}}[-\theta'(\eta)], \tag{16}$$

where the value of $-\theta'(\eta)$ is plotted in Fig. 3, which shows that its value decreases from a maximum value to zero as η is increased from zero. The expression for surface heat flux $q(x, 0)$ is given by equation (16) with $\eta = 0$ and the total surface heat rate (per unit width perpendicular to the x-y plane) along the plane source or sink with a height L is

$$Q = \int_0^L q(x, 0)\,dx = kA^{3/2}\left[\frac{\rho_\infty g\beta K}{\mu\alpha}\right]^{1/2} \times\left(\frac{2}{1+3\lambda}\right)L^{\frac{1+3\lambda}{2}}[-\theta'(0)], \tag{17}$$

where the value of $[-\theta'(0)]$ for selected values of λ is tabulated in Table 1 and is plotted in Fig. 6, which shows that heat-transfer rate increases as the value of f_w is increased. Consequently, the values of $q(x)$ and Q increase as the value of f_w is increased.

Table 1. Values of $-\theta'(0)$ and η_δ for the cases of uniform wall temperature distribution ($\lambda = 0$), uniform surface heat flux ($\lambda = 1/3$) and uniform wall velocity distribution ($\lambda = 1$)

	$-\theta'(0)$			η_δ		
f_w	$\lambda = 0$	$\lambda = 1/3$	$\lambda = 1$	$\lambda = 0$	$\lambda = 1/3$	$\lambda = 1$
−1.0	0.2043	0.3971	0.6180	7.84	7.58	7.20
−0.8	0.2432	0.4416	0.6770	7.58	7.21	6.67
−0.6	0.2865	0.4917	0.7440	7.29	6.81	6.12
−0.4	0.3345	0.5476	0.8198	6.98	6.40	5.59
−0.2	0.3870	0.6096	0.9049	6.65	5.98	5.08
0	0.4438	0.6776	1.000	6.31	5.57	4.60
0.2	0.5050	0.7517	1.104	5.96	5.17	4.16
0.4	0.5701	0.8316	1.219	5.61	4.79	3.77
0.6	0.6389	0.9169	1.344	5.28	4.44	3.42
0.8	0.7111	1.007	1.477	4.96	4.11	3.12
1.0	0.7863	1.102	1.618	4.65	3.81	2.85

With the aid of equations (13) and (8b), the horizontal velocity component at infinity is given by

$$v_\infty = -\frac{1}{2}\left[\frac{\alpha\rho_\infty g\beta K|T_w - T_\infty|}{\mu x}\right]^{1/2}(1+\lambda)f(\infty), \tag{18}$$

which can be positive or negative depending on the sign of $f(\infty)$, which in turn, depending on the value of f_w. Although $f(\infty)$ is positive for the range of parameters considered, as is shown in Fig. 4, it could be negative for sufficiently large negative values of f_w, i.e. for strong discharge rates.

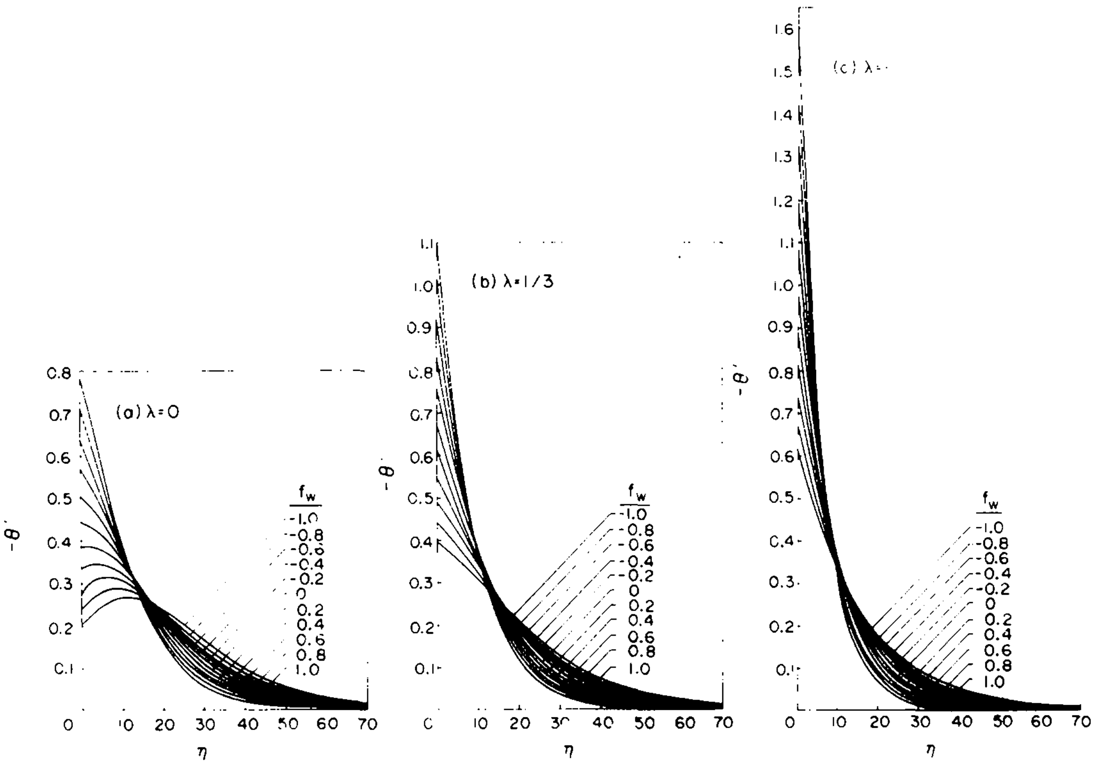

FIG. 3. Values of $-\theta'$ vs η: (a) Uniform wall temperature distribution ($\lambda = 0$), (b) Uniform surface heat flux ($\lambda = 1/3$), and (c) Uniform wall velocity distribution ($\lambda = 1$).

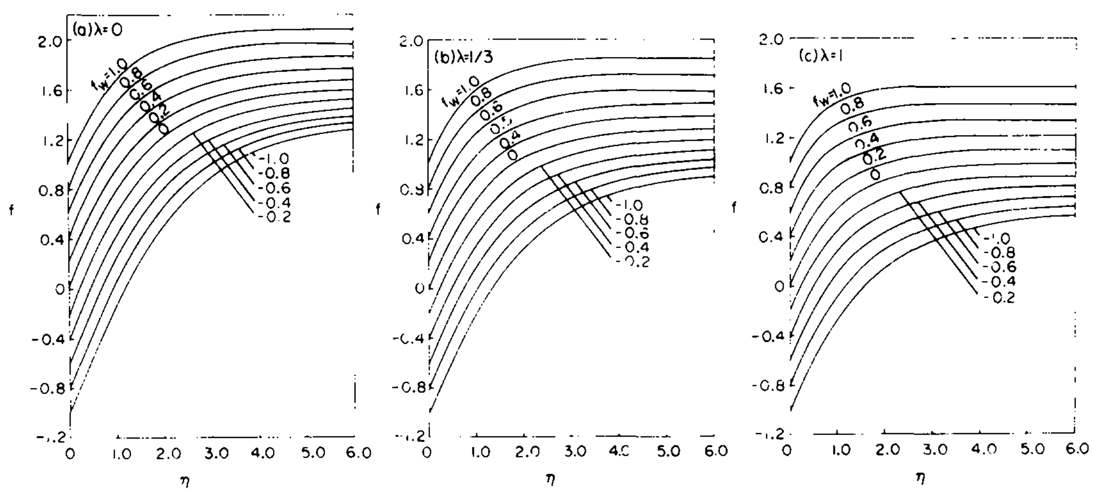

FIG. 4. Values of f vs η: (a) Uniform wall temperature distribution ($\lambda = 0$), (b) Uniform surface heat flux ($\lambda = 1/3$), and (c) Uniform wall velocity distribution ($\lambda = 1$).

Finally, the total mass-transfer rate (per unit width perpendicular to the x–y plane) along a vertical plane source or sink with a length L is

$$\dot{m} = \rho_\infty \int_0^L v(x, 0)\,\mathrm{d}x = 2\rho_\infty a L^{\frac{\lambda+1}{2}}/(1+\lambda), \quad (19)$$

where we have made use of equation (3b).

As is discussed in [6], the range of λ for which the problem is physically realistic is $0 \leqslant \lambda \leqslant 1$ which follows from the simultaneous consideration of equations (13) and (15). We shall now discuss the variation of δ and $q(x, 0)$ as given by equations (15) and (16), for the special cases of $\lambda = 0$, $1/3$, and 1.

(a) $\lambda = 0$ corresponding to the case of uniform wall temperature with $v_w \sim x^{-1/2}$, $\delta \sim x^{1/2}$ and $q \sim x^{-1/2}$.

(b) $\lambda = 1/3$ corresponding to the case of uniform heat flux with $v_w \sim x^{-1/3}$, $T_w \sim x^{1/3}$ and $\delta \sim x^{1/3}$.

(c) $\lambda = 1$ corresponding to the case of uniform wall velocity with $T_w \sim x$, δ = constant and $q \sim x$.

To gain some insight on the magnitude of various physical quantities, consider the discharge of warm geothermal water at 90°C (i.e. $\lambda = 0$) from a fissure crack or well of 500 m to an aquifer at 15°C with $f_w = 1.0$. We used the following value of physical properties for computations: $\rho_\infty = 0.92 \times 10^6$ g/m^3, $C_p =$

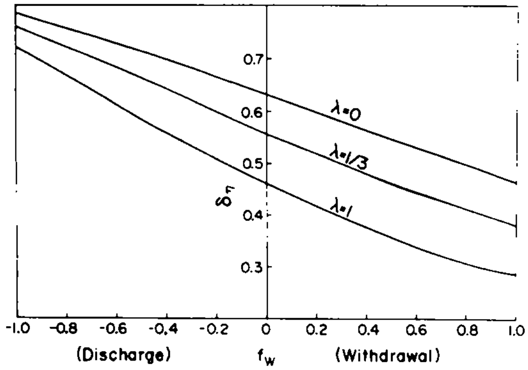

FIG. 5. Effect of mass transfer on boundary-layer thickness.

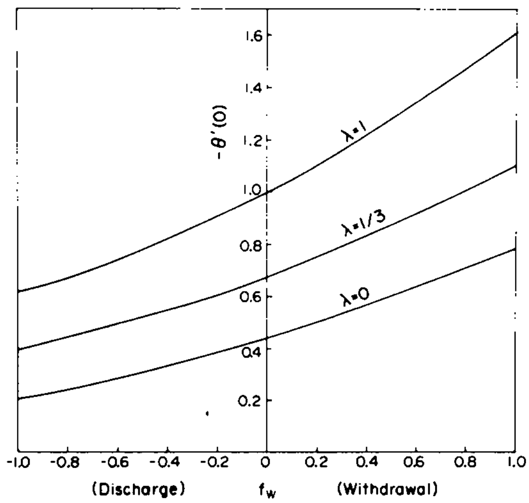

FIG. 6. Effect of mass transfer on surface heat-transfer rate.

1 cal/g°C, $k = 0.58$ cal/s°C m, $\beta = 2.8 \times 10^{-4}$/°K, $g = 9.8$ m/s^2, $\mu = 0.68$ g/s m and $K = 10^{-10}$ m^2. From the definition of f_w and equation (19), we found that the discharge rate is approximately 45 gal/h per meter width perpendicular to the x–y plane. The corresponding boundary-layer thickness as given by equation (15) is approximately 30 m at $x = 500$ m.

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INFLUENCE DU FLUX MASSIQUE SUR LES COUCHES LIMITES DE CONVECTION NATURELLE DANS UN MILIEU POREUX ET SATURE

Résumé—On étudie analytiquement les effets d'un flux massique latéral à température et vitesse données, sur des couches limites de convection naturelle dans un milieu poreux saturé, à des nombres de Rayleigh importants. Dans le cadre de la théorie de la couche limite, des solutions de similitude sont obtenues dans le cas spécial où la température et la vitesse du fluide varient respectivement comme x^{λ} et $x^{(\lambda-1)/2}$. On montre l'effet du flux massique sur le transfert thermique pariétal. On discute l'application à la décharge d'eau chaude le long d'un puits ou d'une fissure d'un aquifère d'étendue infinie.

DER EINFLUSS EINES LATERALEN MASSENAUSFLUSSES AUF DIE GRENZSCHICHTEN BEI FREIER KONVEKTION IN EINEM GESÄTTIGTEN PORÖSEN MEDIUM

Zusammenfassung—Es wird der Einfluß eines lateralen Massenausflusses mit vorgeschriebener Temperatur und Geschwindigkeit auf die Grenzschichten bei freier Konvektion in einem gesättigten porösen Medium bei hohen Rayleigh-Zahlen analytisch untersucht. Für den speziellen Fall, daß sich Temperatur und Geschwindigkeit des Fluids mit x^{λ} bzw. $x^{(\lambda-1)/2}$ verändern, werden Aehnlichkeitslösungen im Rahmen der Grenzschichttheorie abgeleitet. Der Einfluß des Massenausflusses auf den Wärmeübergang an der Oberfläche und die Grenzschichtdicke wird aufgezeigt. Die Anwendung auf die Warmwasserabgabe längs einer Bohrung oder eines Spaltes in ein unendlich ausgedehntes Aquifer wird diskutiert.

ВЛИЯНИЕ ПОПЕРЕЧНОГО ИСТЕЧЕНИЯ МАССЫ НА ПОГРАНИЧНЫЕ СЛОИ ПРИ СВОБОДНОЙ КОНВЕКЦИИ В НАСЫЩЕННОЙ ПОРИСТОЙ СРЕДЕ

Аннотация — Проведено теоретическое изучение влияния поперечного потока массы при заданной температуре и скорости на вертикальные пограничные слои в насыщенной пористой среде при наличии свободной конвекции и больших числах Релея. В рамках теории пограничного слоя получены автомодельные решения для специального случая, когда заданная температура и скорость жидкости изменяются, соответственно, как x^{λ} и $x^{(\lambda-1)/2}$. Показано влияние потока массы на интенсивность поверхностного теплообмена и толщину пограничного слоя. Рассматривается случай истечения нагретой воды по скважине или щели в бесконечный пласт.